

Gravitational Effects of Earth in Optimizing ΔV for Deflecting Earth-Crossing Asteroids

I. Michael Ross,^{*} Sang-Young Park,[†] and Scott D. V. Porter[‡]
Naval Postgraduate School, Monterey, California 93943

Analyses incorporating the gravitational effects of Earth to calculate optimal impulses for deflecting Earth-crossing asteroids are presented. The patched conic method is used to formulate the constrained optimization problem. Geocentric constraints are mapped to heliocentric variables by the use of the impact parameter. The result is a unified nonlinear programming problem in the sense that no distinctions are made for short or long warning times. Numerical solutions indicate that the ΔV requirements are considerably more than those of the previously published two-body analysis that excluded third-body effects. Generally speaking, the increments in the minimum ΔV due to the gravitational effects of the Earth are large (by as much as 60%) for near-Earth asteroids, and the errors diminish for orbits with large eccentricities ($e > 0.7$). Some interesting results for short warning times are also discussed.

Nomenclature

F, G	=	Lagrange coefficients for position
F_t, G_t	=	Lagrange coefficients for velocity
J	=	objective function for nonlinear programming problem
R, R	=	radius vector from Earth to Earth-crossing asteroid (ECA), its magnitude
R_b	=	impact radius and its generalization
R_{critical}	=	proposed miss distance between Earth and ECA
R_{SOI}	=	radius of Earth's sphere of influence
R_{\oplus}	=	radius of Earth
\mathbf{r}_a^0	=	initial heliocentric position vector of ECA
t_b	=	time when $R(t) = R_b$
t_f	=	time of closest approach between Earth and asteroid after application of ΔV
$t_{\Delta V}$	=	time of application of ΔV
t_1	=	time when an ECA impacts Earth
\mathbf{V}_a^0	=	initial inertial velocity of ECA
\mathbf{V}_{∞}	=	hyperbolic excess velocity
$\mathbf{V}_{\oplus}, \mathbf{V}_a$	=	inertial velocities of Earth and asteroid, respectively
$\Delta \mathbf{V}$	=	impulsive velocity increment imparted to an ECA
$\ \Delta \mathbf{V}\ $	=	magnitude of $\Delta \mathbf{V}$
μ_{\oplus}	=	gravitational constant of Earth

Introduction

FOLLOWING the 1980 report of Alvarez et al.,¹ NASA and the Jet Propulsion Laboratory (JPL) cosponsored a 1981 workshop² with the goal of determining the probability and risk of impacts between near-Earth objects and Earth. Since then, there have been many workshops to study the fundamentals of the impact and impact mitigation problem.^{3,4} Much of the concentration of the workshops have been related to the systems analysis of the impact problem, assessing the magnitude of the threat, impact effects and hazards to Earth, as well as the political implications of developing an impact mitigation capability. The survey in Ref. 5 shows that only a limited analysis has been performed on the astrodynamics of the impact mitigation problem. Much of the previous analyses^{5–8} has been limited to analytical results obtained by approximations. One clear conclusion from these simplified analyses is that early detection gives longer reaction time, and asteroid interceptions far

from Earth are much more desirable and easier than interceptions near Earth. This simply follows from the fact that small deflections far away will produce a greater miss distance at Earth. Recent numerical analyses show that there is much to be gained in analyzing the optimization problem.^{9,10} This is primarily because of the coupling between warning times, available energy for deflection, and the sensitivity of the optimization problem.

A number of dynamics and control problems in hazard mitigation are defined in Ref. 5. One of these problems included an outline of an optimization problem as a nonlinear programming problem (NLP). This NLP formulation was furthered in Ref. 9, yielding some interesting results. For example, a slightly off-parallel impulse at perihelion will yield a higher miss distance than a parallel one. In this paper, we extend our previous heliocentric two-body analysis⁹ to include the gravitational effects of Earth. This is achieved by simply mapping the geocentric constraints to heliocentric variables. The result is an NLP formed as an outgrowth of a patched-conic approximation. To distinguish this problem formulation from our previous two-body formulation,⁹ we occasionally refer to it as the three-body model.

Our analysis centers on how optimal impulses applied to an asteroid at various points on the asteroid's orbit affect the outcome when there is a presumption of collision otherwise. The analysis tool presented may be utilized in determining a reasonably accurate estimate for optimizing the time and position of intercepting the asteroid for impact mitigation. Because our problem formulation does not really distinguish between short or long warning times, it represents a more unified approach to the deflection problem. We do, however, distinguish the results for short and long warning times because the differences between them are quite substantial.

Formulation of the Optimization Problems

General Problem

Given an Earth-crossing asteroid (ECA) with an orbit that confirms an impending collision with Earth, the problem is to find the minimum ΔV to prevent a collision. Mathematically, the problem is posed as follows (see Fig. 1): minimize

$$J = \|\Delta \mathbf{V}\| \quad (1)$$

such that

$$\min_{t \geq t_{\Delta V}} R(\Delta \mathbf{V}, t) \geq R_{\text{critical}} \quad (2)$$

In Eq. (2), R is written as $R(\Delta \mathbf{V}, t)$ to emphasize its fundamental functional relationship as it pertains to the problem at hand. Thus, the requirement of an impending collision simply means that $R(\mathbf{0}, t_1) < R_{\text{critical}}$ for some t_1 . In much of the literature, R_{critical} is the radius of Earth (~ 6378 km). For notational convenience and other

Received 21 January 2000; revision received 23 April 2001; accepted for publication 2 May 2001. This material is declared a work of the U.S. Government and is not subject to copyright protection in the United States.

^{*}Associate Professor, Department of Aeronautics and Astronautics; currently on sabbatical at Charles Stark Draper Laboratory, Inc., Cambridge, MA 02139. Associate Fellow AIAA.

[†]Postdoctoral Associate, Department of Aeronautics and Astronautics. Senior Member AIAA.

[‡]Graduate Student, Department of Aeronautics and Astronautics.

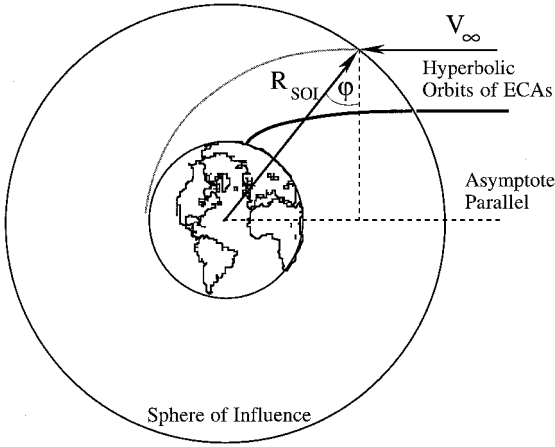


Fig. 2 Definition of angle φ at SOI (not to scale).

$J = \|\Delta V\|$ subject to the constraints given by Eqs. (11) and (12) [or, alternatively, Eq. (19)]. In Eq. (11), $\|\mathbf{R}(\Delta V, t_b)\|$ is computed from

$$\mathbf{R}(\Delta V, t_b) \equiv \mathbf{r}_a^0 F(t_b) + (\mathbf{V}_a^0 + \Delta \mathbf{V}) G(t_b) - \mathbf{r}_\oplus(t_b) \quad (20)$$

and $R_b(t_b)$ is determined from Eqs. (15) and (16), where $\mathbf{V}_a(\Delta V, t)$ is determined from

$$\mathbf{V}_a(\Delta V, t_b) = \mathbf{r}_a^0 F_t(t_b) + (\mathbf{V}_a^0 + \Delta \mathbf{V}) G_t(t_b) \quad (21)$$

where F_t and G_t are the well-known Lagrange coefficients (see Ref. 12) that correspond to the components of velocity at time t_b in $(\mathbf{r}_a^0, \mathbf{V}_a^0)$ basis.

SOI-Based Patched-Conic Formulation

An alternative to the preceding method is to continue to use a static critical radius and to patch the conics at Earth's SOI. In principle, this is a more accurate method. In this method,⁵ the problem is subject to the heliocentric Keplerian equations outside the Earth's SOI and geocentric two-body equations inside the SOI. Figure 2 shows the geometry of this approach for the two-dimensional case. Under this framework, the constraints for the optimization problem can be described in terms of the terminal boundary conditions at the time t_{SOI} when the asteroid intersects the Earth's SOI. These constraints are given by

$$R - R_{\text{SOI}} = 0 \quad (22)$$

$$R_{\text{SOI}} \cos \varphi - R_b(t_{\text{SOI}}) = 0 \quad (23)$$

$$\dot{R} < 0 \quad (24)$$

where φ is given by (see Fig. 2)

$$\cos(\varphi + 90^\circ) = \frac{V_\infty R_{\text{SOI}}}{V_\infty \cdot R_{\text{SOI}}} \quad (25)$$

Here, V_∞ is determined from Eq. (16), but at the point t_{SOI} . \dot{R} is computed from the usual two-body dynamic equations.⁹ Because the radius of the Earth's SOI is about 9.0×10^5 km, or, equivalently, about 0.0060 astronomical units (AU), it is apparent that this patched conic method (patching the conic at $R = 0.0060$) should produce nearly the same result as the one described earlier (patching the conic at $R = 0$).

Numerical Analysis

Because of the enormous number of possibilities between the various parameters, we limit the scope of the numerical analysis to coplanar orbits. As expected, the two NLPs posed in the preceding section yield nearly the same results. The MATLAB[®] optimization toolbox¹³ was used to solve the NLPs. For well-documented reasons and numerical scaling, heliocentric canonical units were used. In this system, the distance units (DU), time units (TU), and speed units (SU) are 1 DU = 1 AU = 1.49596×10^8 km, 1 TU = $1/2\pi$ year = 58.17 days, and 1 SU = 29.80 km/s. In the discussions to follow, we will

use the term impulse time to mean specifically either $(t_1 - t_{\Delta V})$ or its absolute value; the exact usage will be obvious from the context. Without loss in generality, we choose $t_1 = 0$. This initialization has the advantage of interpreting $t_{\Delta V}$ as the time interval before impact if no action, that is, ΔV maneuver, is undertaken. Also, because the warning time (i.e., the time interval between detection and collision) must be greater than the impulse time, the latter provides a necessary lower bound for the former.

Note the sensitivity of the optimization algorithm to initial guesses in the numerical solution of the optimization problem. The sensitivity reduction procedure described in Ref. 9 for the two-body problem works with little modification for the current problem formulation as well. To conform to some of the analysis established in the literature, we distinguish the results between long and short warning times. In addition, we choose $R_{\text{critical}} = 1$ Earth radius (4.263×10^{-5} DU).

Analysis for Long Warning Times

Long warning times loosely correspond to $t_{\Delta V}$ greater than one orbital period of the asteroid. Consider a fictitious asteroid whose orbital elements are given by $a = 1.5$ AU (period of about 1.85 years) and $e = 0.5$. In Figs. 3 and 4, the magnitudes and angles of the optimal impulses are compared with those from Ref. 9 that excluded the gravitational effects of Earth. The impulse time is normalized to the period of the unperturbed asteroid for ease of interpretation. Not surprisingly, the minimum required ΔV increases dramatically as t approaches 0. As expected, the minimum ΔV is larger when the gravitational effects of the Earth are included. However, what is noteworthy is that the three-body requirements (i.e., the ones obtained by the patched-conic approximations) are significantly higher than those obtained from two-body approximations, that is, ignoring the gravity of Earth. For example, at the second perihelion point (which is at $t = -2.0746$ asteroid's orbital periods), the two-body ΔV requirement is 0.8694 cm/s, whereas

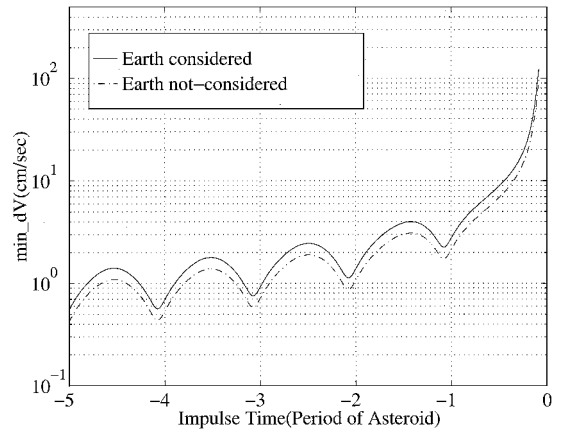


Fig. 3 Minimum ΔV vs impulse time.

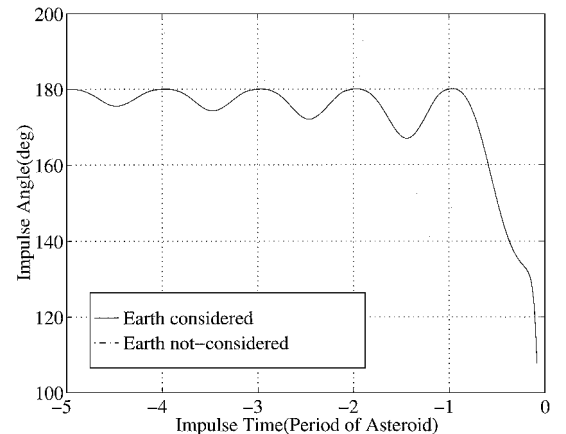


Fig. 4 Optimal impulse angle vs impulse time.

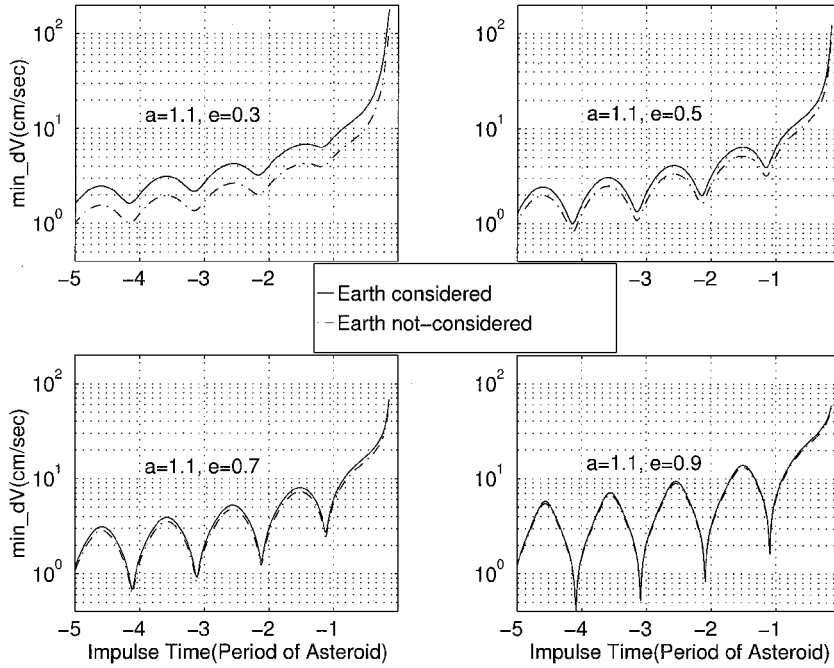


Fig. 5 Minimum ΔV with Earth's gravitational effects for various eccentricities.

the three-body requirement jumps up by 29.7% to 1.1275 cm/s. Figure 4 shows a corresponding plot of the optimal impulse angle with respect to impulse time. The impulse angle is defined as the angle from the ECA's original velocity vector to the impulse vector through the sunward direction. It is apparent from Fig. 4 that the optimal angles for both the two-body and three-body cases are identical. In other words, the effect of Earth's gravity simply increases the magnitude of the minimum ΔV (leaving the angle unchanged). This can be attributed to that, because the ΔV requirements are quite small, it has a linear relationship to the miss distance. Because Earth's gravity has the effect of essentially enlarging the miss distance, the optimal impulse angle should really be unchanged. Thus, Fig. 4 confirms this observation and, in some way, validates our methods.

The effects of Earth's gravity can have quite a dramatic effect on near-Earth asteroids on a collision course with Earth. Figure 5 shows the histories of the minimum ΔV for various fictitious asteroids whose orbital elements are given by $a = 1.1$ AU, $e = 0.3, 0.5, 0.7$, and 0.9 . The plots clearly show that the two-body assumptions do not hold for near-Earth ECAs, particularly for those that have lower eccentricities. For example, the minimum ΔV obtained by adding the effects of the third body (Earth) is 59.6% larger than the two-body solution at the second perihelion for the case of $a = 1.1$ AU and $e = 0.3$ (first subfigure in Fig. 5). The large differences are a direct consequence of the increasing value of the impact parameter for slower and, hence, nearer-Earth asteroids.

To confirm the results from the optimization algorithm, a simple two-body propagation code was developed for geocentric hyperbolic trajectories. The idea is to use the converged results from the optimization code as initial conditions for propagating the position of the asteroid inside the SOI. Thus, \mathbf{R}_{SOI} , \mathbf{V}_{∞} , and t_{SOI} form the initial conditions for the propagation. The propagation was performed by the method of Lagrangian coefficients for hyperbolic orbits (see Ref. 12). The results of this propagation for the case corresponding to Fig. 1, that is, $a = 1.5$ and $e = 0.5$, are displayed in Figs. 6 and 7. The unperturbed trajectory is on a rectilinear collision course with Earth. The asteroid perturbed by two-body assumptions still collides with Earth, but with a reduced impact angle. The trajectory obtained from the patched-conic concept (or three-body approximation) provides sufficient ΔV for the asteroid to miss Earth (by the specified amount of one Earth radius).

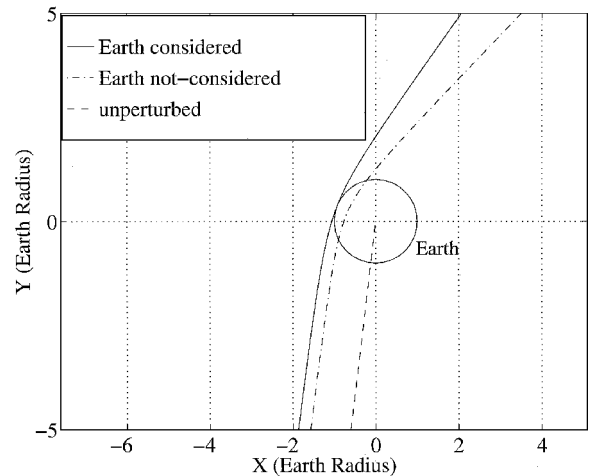


Fig. 6 Hyperbolic trajectories with respect to Earth.

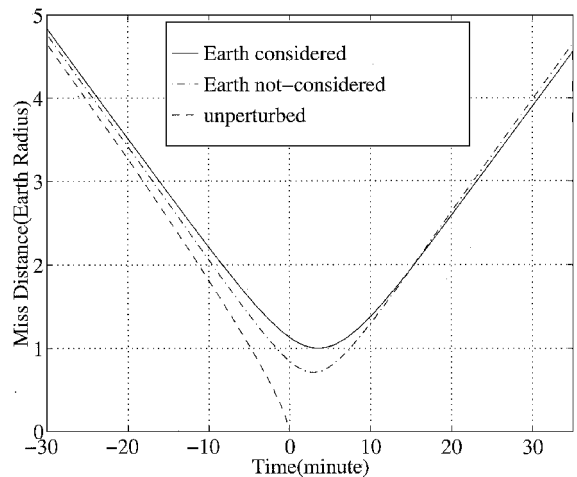


Fig. 7 Miss distances from Earth.

Analysis for Short Warning Times

Short warning times loosely correspond to $t_{\Delta V}$ less than one orbital period of the asteroid. In previous studies^{6,14,15} the Earth's gravitational effects were ignored, and a rectilinear motion was assumed. The following simple computation underscores the effect of this assumption. If a ΔV is imparted to an asteroid for missing Earth by R_{critical} , then the trajectory is bent toward Earth by the perigee of a hyperbolic orbit,

$$r_p = -(\mu_{\oplus}/V_{\infty}) + \sqrt{(\mu_{\oplus}/V_{\infty}^2) + R_{\text{critical}}^2} \quad (26)$$

Using R_{critical} of one Earth-radius and a typical impact velocity of 20 km/s, we get $r_p = 0.856$ of Earth radius. Thus, preliminary approximations of energy required to deflect a threat may result in an error of 14% in miss distance.

Consider once again the fictitious asteroids characterized by a semimajor axis of $a = 1.1$ (orbital period ≈ 1.15 years), and eccentricities given by $e = 0.3, 0.5, 0.7$, and 0.9 . Figures 8 and 9 display the optimal impulse angles and minimum ΔV required for impulse times of up to about 15 days before impact. It is apparent from Figs. 8 and 9 that the larger the orbit eccentricity, the greater the fluctuations in both the minimum ΔV and the optimal impulse angle. What is more interesting is that the optimal impulse angles are not perpendicular to the velocity but are instead about 40 deg for the specific examples considered here. Note, however, that the optimal angles are measured from the ECA's original heliocentric velocity and not from the ECA's velocity with respect to the Earth. Another interesting feature of the results is that the optimal ΔV appear to

Table 1 Position and velocity of asteroids ($a = 1.1$) at Earth's SOI

Parameter	$e = 0.3$	$e = 0.5$	$e = 0.7$	$e = 0.9$
Impact parameter b_i	$1.601 R_{\oplus}$	$1.230 R_{\oplus}$	$1.112 R_{\oplus}$	$1.058 R_{\oplus}$
X , km	-11,700	175,000	308,200	479,300
Y , km	-928,900	-912,300	-876,300	-795,700
V_X , km/s	0.01	-2.82	-7.78	-16.87
V_Y , km/s	8.94	15.35	21.59	27.56

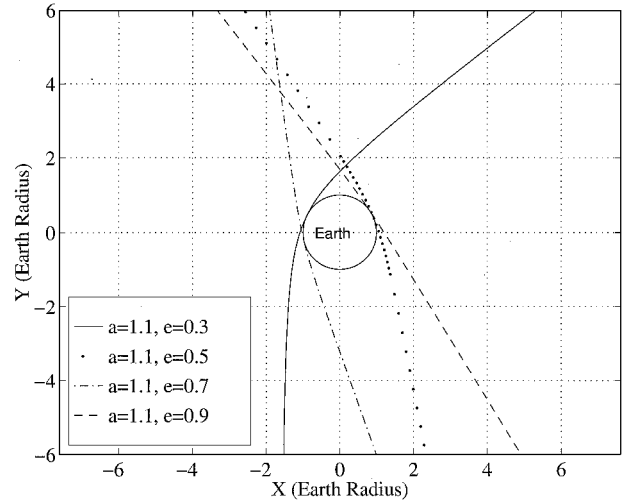


Fig. 10 Trajectories of asteroids at near Earth for short warning times.

target a specific impact parameter. For a given orbit of the asteroid, the optimal trajectories yielded nearly the same position vector, velocity vector, and impact parameter for all impulse times (of less than about 15 days). Table 1 shows the converged values for the cases considered. To confirm these results, the same propagation technique adopted for the long warning times were used. Figure 10 shows the propagated hyperbolic trajectories for the position and velocity vectors given in Table 1. The asteroid misses the Earth by the specified amount of one Earth radius in all of the cases, thus validating (to some degree) the optimality of the results.

Conclusions

The minimum ΔV required to deflect an ECA is substantially greater than previous estimates established elsewhere, particularly for nearer-Earth asteroids, that is, those asteroids whose semimajor axes are near 1 AU. This is a direct consequence of the impact radius having an inverse relationship to the hyperbolic excess velocity: The nearer and slower asteroids are more influenced by the Earth's gravity than the farther and faster ones. The increase in the ΔV due to the gravitational effects of Earth may be quickly estimated from a rule of thumb: The ΔV obtained from the Keplerian two-body model underestimates the true ΔV by approximately the impact parameter normalized by the miss distance $\{\approx \sqrt{[1 + 2\mu_{\oplus}/R_{\oplus}V_{\infty}^2(t)]}\}$.

Acknowledgments

Funding for this research was provided, in part, by the Air Force Space Command and the Naval Postgraduate School (NPS). We would like to thank Brian Neuenfeldt for sponsoring this research and Paul Penzo of the Jet Propulsion Laboratory, California Institute of Technology, for suggesting the use of the impact parameter to modify our original analysis. This work was performed while the second author was a National Research Council Postdoctoral Associate at NPS. A version of this paper (AAS 98-186) was presented at the AAS/AIAA Spaceflight Mechanics Meeting, Monterey, CA, 9–11 February 1998.

References

- Alvarez, L. W., Alvarez, W., Asaro, F., and Michel, H., "Extra-Terrestrial Cause for the Cretaceous-Tertiary Extinction," *Science*, Vol. 208, 1980, pp. 1095–1108.
- Collision of Asteroids and Comets with the Earth: Physical and Human Consequences, Snowmass, CO, 1981.

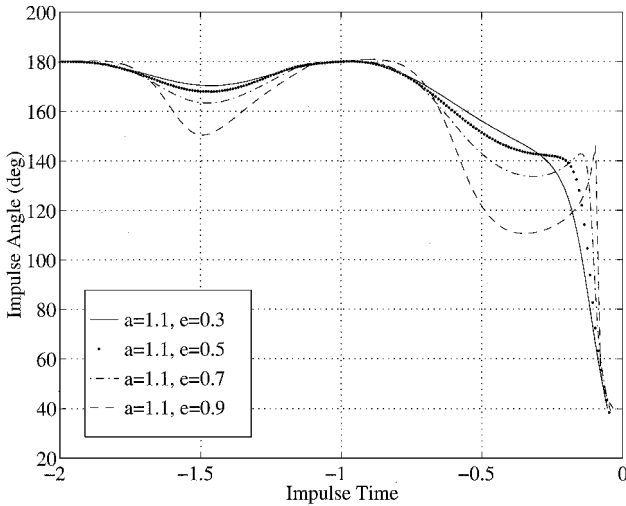


Fig. 8 Histories of optimal impulse angles for short warning times.

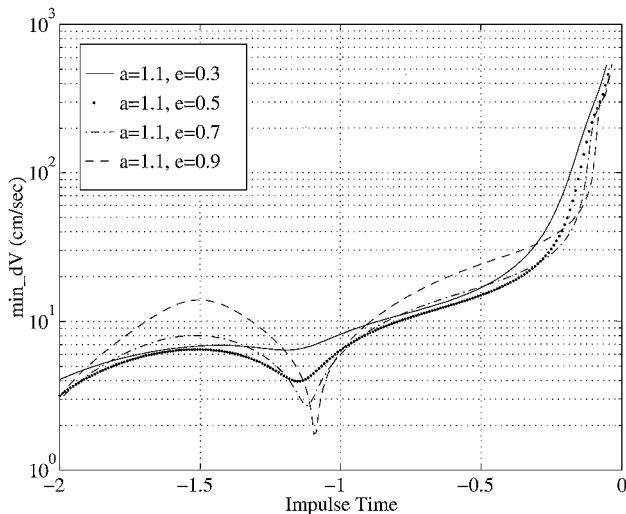


Fig. 9 Histories of minimum ΔV for short warning times.

³Canavan, G. H., Solem, J. C., and Rather, J. D. G., (eds.), *Proceedings of the Near-Earth-Object Interception Workshop*, Los Alamos National Lab., Los Alamos, NM, 1992.

⁴Gehrels, T., (ed.), *Hazards due to Comets and Asteroids*, Univ. of Arizona Press, Tucson, AZ, 1994.

⁵Hall, C. D., and Ross, I. M., "Dynamics and Control Problems in the Deflection of Near-Earth Objects," *Advances in the Astronautical Sciences, Astrodynamics 1997*, Vol. 97, Pt. 1, 1997, pp. 613–631.

⁶Ahrens, T. J., and Harris, A. W., "Deflection and Fragmentation of Near-Earth Asteroids," *Nature*, Vol. 360, 1992, pp. 429–433.

⁷Solem, J. C., "Interception of Comets and Asteroids on Collision Course with Earth," *Journal of Spacecraft and Rockets*, Vol. 30, No. 2, 1993, pp. 222–228.

⁸Ivashkin, V. V., and Smirnov, V. V., "An Analysis of Some Methods of Asteroid Hazard Mitigation for the Earth," *Journal of Planetary and Space Science*, Vol. 43, No. 6, 1995, pp. 821–825.

⁹Park, S.-Y., and Ross, I. M., "Two-Body Optimization for Deflecting Earth-Crossing Asteroids," *Journal of Guidance, Control, and Dynamics*, Vol. 22, No. 3, 1999, pp. 415–420.

¹⁰Elder, J. T., "Optimal Impulse Conditions for Deflecting Earth-Crossing Asteroids," M.S. Thesis, Dept. of Aeronautics and Astronautics, Naval Postgraduate School, Monterey, CA, Dec. 1997.

¹¹Brown, C., *Spacecraft Mission Design*, AIAA, Washington, DC, 1992, pp. 27, 117.

¹²Battin, R. H., *Introduction to the Mathematics and Methods of Astrodynamics*, AIAA, New York, 1987, pp. 153–178.

¹³Grace, A., *Optimization TOOLBOX User's Guide*, MathWorks, Inc., Natick, MA, 1992, pp. 1.13–3.12.

¹⁴Solem, J. C., and Snell, C. M., "Terminal Intercept for Less than One Orbital Period Warning," *Hazards due to Comets and Asteroids*, edited by T. Gehrels, Univ. of Arizona Press, Tucson, AZ, 1994, pp. 1013–1034.

¹⁵Meissinger, H. F., "Technology Assessment for Defense Against Asteroids or Comet," *Proceedings of the Planetary Defense Workshop* [online], URL: <http://www.llnl.gov/tid/lof/documents/toc/232015.html> [cited 28 July 1999].

D. B. Spencer
Associate Editor